Title	Accumulation of periodic points for local uniformly quasiregular mappings (Potential Theory and its Related Fields)
Author(s)	OKUYAMA, Yusuke; PANKKA, Pekka
Citation	数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2013), B43: 121-139
Issue Date	2013-09
URL	http://hdl.handle.net/2433/209063
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

RIMS Kôkyûroku Bessatsu
B43 (2013), 121–139

Accumulation of periodic points for local uniformly quasiregular mappings

By

Yûsuke OKUYAMA* and Pekka PANKKA**

Abstract

We consider accumulation of periodic points in local uniformly quasiregular dynamics. Given a local uniformly quasiregular mapping f with a countable and closed set of isolated essential singularities and their accumulation points on a closed Riemannian manifold, we show that points in the Julia set are accumulated by periodic points. If, in addition, the Fatou set is non-empty and connected, the accumulation is by periodic points in the Julia set itself. We also give sufficient conditions for the density of repelling periodic points.

§ 1. Introduction

Let M and N be oriented Riemannian n-manifolds for $n \geq 2$. A continuous mapping $f\colon M \to N$ is called *K-quasiregular*, $K \geq 1$, if f belongs to the Sobolev space $W^{1,n}_{\mathrm{loc}}(M,N)$ and satisfies the distortion inequality

$$\|df\|^n \leq K J_f \quad \text{a.e. on } M,$$

where $\|df\|$ is the operator norm of the differential df of f and J_f the *Jacobian determinant* of f satisfying $f^*(\mathrm{vol}_N) = J_f \mathrm{vol}_M$, where vol_M and vol_N are the Riemannian volume forms on M and N, respectively.

A quasiregular self-map $f\colon M \to M$ is called *uniformly K-quasiregular (K-UQR)* if all iterates f^k for $k \geq 1$ are K-quasiregular. Similarly as quasiregular mappings have

Received January 31, 2013. Revised July 7, 2013.
2000 Mathematics Subject Classification(s): Primary 30C65; Secondary 37F10, 30D05
Key Words: *local uniformly quasiregular mapping, repelling periodic point, Julia set*
Y. O. is partially supported by JSPS Grant-in-Aid for Young Scientists (B), 24740087.
P. P. is partially supported by the Academy of Finland project #256228.
*Division of Mathematics, Kyoto Institute of Technology, Sakyo-ku, Kyoto 606-8585, Japan.
e-mail: `okuyama@kit.ac.jp`
**University of Helsinki, Department of Mathematics and Statistics (P.O. Box 68), FI-00014 University of Helsinki, Finland.
e-mail: `pekka.pankka@helsinki.fi`

the rôle of holomorphic mappings in the n-dimensional Euclidean conformal geometry for $n \geq 3$, the dynamics of uniformly quasiregular mappings can be viewed as the counterpart of holomorphic dynamics in the n-dimensional conformal geometry. We refer to the seminal paper of Iwaniec and Martin [12] and Hinkkanen, Martin, Mayer [9] for the fundamentals in this theory.

In this article we consider dynamics of local UQR-mappings. Let M be an oriented Riemannian n-manifold and $\Omega \subset M$ an open set. Following the terminology in [9], we say a mapping $f\colon \Omega \to M$ is a *local uniformly K-quasiregular*, $K \geq 1$, if for every $k \in \mathbb{N}$, $\bigcap_{j=0}^{k-1} f^{-j}(\Omega) \neq \emptyset$ and $f^k\colon \bigcap_{j=0}^{k-1} f^{-j}(\Omega) \to M$ is K-quasiregular.

With slight modifications, the standard terminology from dynamics is at our disposal also in this local setting. Let

$$D_f := \text{the interior of } \bigcap_{k\geq 0} f^{-k}(\Omega) = M \setminus \overline{\bigcup_{k\geq 0} f^{-k}(M\setminus\Omega)}.$$

As usual, the Fatou set $F(f)$ of f is the maximal open subset in D_f where the family $\{f^k; k \in \mathbb{N}\}$ is normal, the Julia set of f is the set

$$J(f) := M \setminus F(f),$$

and the exceptional set of f is

$$\mathcal{E}(f) := \{x \in M; \# \bigcup_{k\geq 0} f^{-k}(x) < \infty\}.$$

A point $x \in M$ is a *periodic point of f in M* if $x \in \bigcap_{j=0}^{p-1} f^{-j}(\Omega)$ and $f^p(x) = x$ for some $p \in \mathbb{N}$. We call p a *period of x (under f)*. Note that periodic points always belong to the set $\overline{D_f}$.

A periodic point $x \in M$ with period $p \in \mathbb{N}$ is *(topologically) repelling* if $f\colon U \to f^p(U)$ is univalent and $U \Subset f^p(U)$ for some open neighborhood U of x in $\bigcap_{j=0}^{p-1} f^{-j}(\Omega)$. Note that, then $x \in J(f)$; see [9, §4].

In [9], Hinkkanen, Martin and Mayer gave a classification of cyclic Fatou components of f (see Theorem 2.12) as well as periodic points. We study both $J(f)$ and $\mathcal{E}(f)$ for a non-constant local uniformly quasiregular mapping

$$f\colon \mathbb{M} \setminus S_f \to \mathbb{M},$$

where $\mathbb{M}$ is a closed, oriented, and connected Riemannian n-manifold, $n \geq 2$, and S_f is a countable and closed subset in $\mathbb{M}$ consisting of isolated essential singularities of f and their accumulation points in $\mathbb{M}$. In our first main theorem, we also consider a subclass of non-elementary UQR-mappings. A non-constant local uniformly quasiregular mapping $f\colon \mathbb{M} \setminus S_f \to \mathbb{M}$ is *non-elementary* if it is non-injective and satisfies

$$J(f) \not\subset \mathcal{E}(f).$$

For comments on the non-injectivity and non-elementarity, see Section 5.

Recall that a point x in a topological space X is *accumulated by* a subset S in X if for every neighborhood N of x, $S \cap (N \setminus \{x\}) \neq \emptyset$, and that a subset S in X is *perfect* if S is non-empty, compact, and has no isolated points in X.

Theorem 1. *Let $\mathbb{M}$ be a closed, oriented, and connected Riemannian n-manifold, $n \geq 2$, and $f\colon \mathbb{M} \setminus S_f \to \mathbb{M}$ a non-constant local uniformly K-quasiregular mapping, $K \geq 1$, where S_f is a countable and closed subset in $\mathbb{M}$ and consists of isolated essential singularities of f and their accumulation points in $\mathbb{M}$. Then $J(f)$ is nowhere dense in $\mathbb{M}$ unless $J(f) = \mathbb{M}$. Furthermore, the following hold:*

(a) If f is non-injective, then $J(f) \neq \emptyset$ and $\#\mathcal{E}(f) < \infty$. Moreover, for every $x \in \mathbb{M} \setminus \mathcal{E}(f)$, points in $J(f)$ are accumulated by $\bigcup_{k\geq 0} f^{-k}(x)$.

(b) If f is non-injective and $S_f = \emptyset$, then $\mathcal{E}(f) \subset F(f)$ and f is non-elementary.

(c) If f is a priori non-elementary, then $J(f)$ is perfect and points in $J(f)$ are accumulated by periodic points of f.

For non-constant and non-injective uniformly quasiregular endomorphisms of the n-sphere $\mathbb{S}^n$, the accumulation of periodic points to $J(f)$ in Theorem 1 is due to Siebert [21, 3.3.6 Theorem]; note that by a theorem of Fletcher and Nicks [6], $J(f)$ is in fact uniformly perfect in this case.

The proof of the accumulation of periodic points to the Julia set for non-elementary f is based on two rescaling principles (see Section 2). It is a generalization of Schwick's argument [19] (see also Bargmann [2] and Berteloot–Duval [3]), which is a reminiscent to Julia's construction of (expanding) homoclinic orbits for rational functions ([14, §14]). Our argument simultaneously treats all the cases $S_f = \emptyset$, $0 < \#\bigcup_{k\geq 0} f^{-k}(S_f) < \infty$, and $\#\bigcup_{k\geq 0} f^{-k}(S_f) = \infty$, which are typically studied separately.

In the final assertion in Theorem 1, it would be natural and desirable to obtain the density of (repelling) periodic points in $J(f)$.

Our second main theorem gives sufficient conditions for those density results. The topological dimension of a subset E in $\mathbb{M}$ is denoted by $\dim E$ and the branch set of f by B_f; the *branch set* B_f is the set of points at which f is not a local homeomorphism.

Theorem 2. *Let $\mathbb{M}$ be a closed, oriented, and connected Riemannian n-manifold, $n \geq 2$, and $f\colon \mathbb{M} \setminus S_f \to \mathbb{M}$ be a non-elementary local uniformly K-quasiregular mapping, $K \geq 1$, where S_f is a countable and closed subset in $\mathbb{M}$ and consists of isolated essential singularities of f and their accumulation points in $\mathbb{M}$. Then*

(a) If $F(f)$ is non-empty and connected, then points in $J(f)$ are accumulated by periodic points of f contained in $J(f)$.

(b) If one of the following four conditions

(i) $\#\bigcup_{k\ge 0} f^{-k}(S_f) < \infty$ *and* $\dim J(f) > n-2$,

(ii) f *has a repelling periodic point in* $D_f \setminus (\mathcal{E}(f) \cup \bigcup_{k\in\mathbb{N}} f^k(B_{f^k}))$,

(iii) $J(f) \not\subset \bigcap_{j\in\mathbb{N}} \overline{\bigcup_{k\ge j} f^k(B_{f^k})}$, *or*

(iv) $n = 2$

holds, then points in $J(f)$ *are accumulated by repelling periodic points of* f.

Theorem 2 combines and extends previous results of Hinkkanen–Martin–Mayer ([9]) and Siebert ([20]) for UQR-mappings and classical results of Fatou and Julia ([14, §14]), Baker [1], Bhattacharyya [4], and Bolsch [5] and Herring [8] in the holomorphic case.

For non-constant and non-injective uniformly quasiregular endomorphisms of $\mathbb{S}^n$, the repelling density in $J(f)$ is due to Hinkkanen, Martin and Mayer [9] when $F(f)$ is either empty or not connected. Under these conditions $S_f = \emptyset$ and $\dim J(f) > n-2$. Siebert [20, 4.3.6 Satz] proved the repelling density under the assumption $J(f) \not\subset \overline{\bigcup_{k\in\mathbb{N}} f^k(B_{f^k})}$. In this case $J(f) \not\subset \bigcap_{j\in\mathbb{N}} \overline{\bigcup_{k\ge j} f^k(B_{f^k})}$.

In the holomorphic dynamics, i.e. for $\mathbb{M} = \mathbb{S}^2$ (so $n = 2$) and $K = 1$, every non-constant and non-injective holomorphic mapping $f\colon \mathbb{S}^2 \setminus S_f \to \mathbb{S}^2$ is non-elementary (see Section 5). For $S_f = \emptyset$, the repelling density in $J(f)$ is a classical result of Fatou and Julia (cf. [14, §14]). For $\#\bigcup_{k\ge 0} f^{-k}(S_f) = 1, 2$ and $\#S_f = \infty$, it is due to Baker [1], Bhattacharyya [4], Bolsch [5] and Herring [8]. Note that our proof covers also the case $\#\bigcup_{k\ge 0} f^{-k}(S_f) > 2$.

This paper is organized as follows. In Section 2, we give a unified treatment for normal families and isolated essential singularities of quasiregular mappings. We also recall the invariance of the dynamical sets $D_f, F(f), J(f)$, and $\mathcal{E}(f)$ under f and the Hinkkanen–Martin–Mayer classification for cyclic Fatou components of non-elementary local uniformly quasiregular mappings. In Sections 3 and 4, we prove Theorems 1 and 2. We finish, in Section 5, with comments on the non-injectivity and non-elementarity of non-constant local uniformly quasiregular dynamics.

§ 2. Preliminaries

We begin with notations and fundamental facts from the local degree theory. For each oriented n-manifold X, we fix a generator ω_X of $H^n_c(X;\mathbb{Z})$ representing the orientation of X, and for each subdomain $D \subset X$, a generator ω_D of $H^n_c(D;\mathbb{Z})$ satisfying $\omega_X = \iota_{D,X}(\omega_D)$, where $\iota_{D,X}\colon H^n_c(D;\mathbb{Z}) \to H^n_c(X;\mathbb{Z})$ is the canonical isomorphism.

Let $f\colon M \to N$ be a continuous mapping between oriented n-manifolds M and N. For each domain $D \subset M$ and each $y \in N \setminus f(\partial D)$, the *local degree of* f *at* $y \in N$ *with* *respect to* D is the non-negative integer $\mu(y, f, D)$ satisfying

$$\mu(y, f, D)\omega_D = \iota_{V,D}((f|V)^*\omega_\Omega), \tag{2.1}$$

where Ω is the component of $N \setminus f(\partial D)$ containing y and $V = f^{-1}(\Omega) \cap D$. Indeed, we can take any open and connected neighborhood of y in $N \setminus f(\partial D)$ as Ω. If $\mu(y, f, D) > 0$, then $f^{-1}(y) \cap D \neq \emptyset$. For more details, see e.g., [7, Section I.2].

From now on, let $n \geq 2$ and $K \geq 1$. Let M and N be connected and oriented Riemannian n-manifolds, and $f \colon M \to N$ a non-constant quasiregular mapping. By Reshetnyak's theorem (see e.g. [18, I.4.1]), f is a *branched cover*, that is, an open and discrete mapping. Every $x \in M$ has a *normal neighborhood* with respect to f, that is, an open neighborhood U of x satisfying $f(\partial U) = \partial(f(U))$ and $f^{-1}(f(x)) \cap U = \{x\}$. We denote by $i(x, f)$ the *topological index of* f *at* x, that is, $i(x, f) = \mu(f(x), f, U)$. The branch set B_f of f is the set of all $x \in M$ satisfying $i(x, f) \geq 2$, and is closed in M. By the Chernavskii-Väisälä theorem [22], the topological dimensions $\dim B_f$ and $\dim f(B_f)$ are at most $n - 2$.

The local degree theory readily yields the following manifold version of the Miniowitz–Rickman argument principle or the Hurwitz-type theorem; see [15, Lemma 2]; note that we do not assume that mappings f_j to be quasiregular.

Lemma 2.1. *Let M and N be oriented Riemannian n-manifolds, $n \geq 2$. Suppose a sequence (f_j) of continuous mappings from M to N tends to a quasiregular mapping $f \colon M \to N$ locally uniformly on M as $j \to \infty$. Then for every domain $D \Subset M$ with $f(\partial D) = \partial(f(D))$ and every compact subset $E \subset N \setminus f(\partial D)$, there exists $j_0 \in \mathbb{N}$ such that $\mu(y, f_j, D) = \mu(y, f, D)$ for every $j \geq j_0$ and every $y \in E$.*

Proof. Let $\Omega \Subset f(D)$ be a domain containing E and set $V := f^{-1}(\Omega) \cap D$. Then $(f|V)^*(\omega_\Omega) \in H^n_c(V; \mathbb{Z})$. Set $V_j := f_j^{-1}(\Omega) \cap D$ for each $j \in \mathbb{N}$. Since $f(\partial D) \cap \Omega = \emptyset$, by the uniform convergence of (f_j) to f on ∂D, there exists $j_0 \in \mathbb{N}$ for which $f_j(\partial D) \cap \Omega = \emptyset$ for every $j \geq j_0$. Thus $(f_j|V_j)^*(\omega_\Omega) \in H^n_c(V_j; \mathbb{Z})$ for $j \geq j_0$. Furthermore, mappings $f|D$ and $f_j|D$ are properly homotopic with respect to Ω for every $j \in \mathbb{N}$ large enough, that is, there exists $j_1 \in \mathbb{N}$ so that for every $j \geq j_1$ there exists a homotopy $F_j \colon \overline{D} \times [0, 1] \to N$ from $f|\overline{D}$ to $f_j|\overline{D}$ and $F_j(\partial D \times [0, 1]) \cap \Omega = \emptyset$. Thus $\iota_{V,D}((f|V)^*\omega_\Omega) = \iota_{V_j,D}((f_j|V_j)^*\omega_\Omega)$ for $j \geq \max\{j_0, j_1\}$, and (2.1) completes the proof. □

A point $x' \in M$ is a *non-normality point* of a family $\mathcal{F}$ of K-quasiregular mappings from M to N if $\mathcal{F}$ is not normal on any open neighborhood of x'. A point $x' \in M$ is an *isolated essential singularity* of a quasiregular mapping $f : M \setminus \{x'\} \to N$ if f does not extend to a continuous mapping from M to N.

From now on, suppose that N is closed. The following theorems are manifold versions Miniowitz's Zalcman-type lemma ([15, Lemma 1]) and a Miniowitz–Zalcman-type rescaling principle for isolated essential singularities, respectively.

Theorem 2.2 ([13, Theorem 19.9.3]). *Let M be an oriented Riemannian n-manifold and N a closed and oriented Riemannian n-manifold, $n \geq 2$, and let $x' \in M$. Then a family $\mathcal{F}$ of K-quasiregular mappings, $K \geq 1$, from M to N is not normal at x' if and only if there exist sequences (x_j), (ρ_j), and (f_j) in $\mathbb{R}^n$, $(0, \infty)$, and $\mathcal{F}$, respectively, and a non-constant K-quasiregular mapping $g\colon \mathbb{R}^n \to N$ such that $\lim_{j\to\infty} x_j = \phi(x')$, $\lim_{j\to\infty} \rho_j = 0$ and*

$$\lim_{j\to\infty} f_j \circ \phi^{-1}(x_j + \rho_j v) = g(v) \tag{2.2}$$

locally uniformly on $\mathbb{R}^n$, where $\phi\colon D \to \mathbb{R}^n$ is a coordinate chart of M at x'.

Theorem 2.3 ([17, Theorem 1]). *Let M be an oriented Riemannian n-manifold and N a closed and oriented Riemannian n-manifold, $n \geq 2$, and let $x' \in M$. Then a K-quasiregular mapping $f : M \setminus \{x'\} \to N$, $K \geq 1$, has an essential singularity at x' if and only if there exist sequences (x_j) and (ρ_j) in $\mathbb{R}^n$ and $(0, \infty)$, respectively, and a non-constant K-quasiregular mapping $g : X \to N$, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, such that $\lim_{j\to\infty} x_j = \phi(x')$, $\lim_{j\to\infty} \rho_j = 0$, and*

$$\lim_{j\to\infty} f \circ \phi^{-1}(x_j + \rho_j v) = g(v) \tag{2.3}$$

locally uniformly on X, where $\phi : D \to \mathbb{R}^n$ is a coordinate chart of M at x'.

By the Holopainen–Rickman Picard-type theorem [10], *for every $n \geq 2$ and every $K \geq 1$, there exists a non-negative integer q such that $\#(N \setminus f(\mathbb{R}^n)) \leq q$ for every closed and oriented Riemannian n-manifold N and every non-constant K-quasiregular mapping $f\colon \mathbb{R}^n \to N$.* We use this Picard-type theorem in this article also in the following form.

Theorem 2.4. *For every $n \geq 2$ and every $K \geq 1$, there exists a non-negative integer q' such that $\#(N \setminus g(X)) \leq q'$ for every closed and oriented Riemannian n-manifold N and every non-constant K-quasiregular mapping $f\colon X \to N$, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$.*

Proof. Let $Z_n\colon \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$ be the Zorich mapping and $K_n \geq 1$ the distortion constant of Z_n; see e.g. [18, I.3.3] for the construction of the Zorich map. Set $K' := K \cdot K_n \geq 1$. Replacing f with $f \circ Z_n$ if necessary, we may assume that f is a K'-quasiregular mapping from $\mathbb{R}^n$ to N. Now the Holopainen–Rickman Picard-type theorem [10] completes the proof. $\square$

Let $q'(n,K)$ be the smallest such $q' \in \mathbb{N} \cup \{0\}$ as in Theorem 2.4, which we call the *quasiregular Picard constant for parameters* $n \geq 2$ *and* $K \geq 1$.

Having a Hurwitz-type theorem (Lemma 2.1) and rescaling theorems for a non-normality point of a family of K-quasiregular mappings and for an essential isolated singularity of a quasiregular mapping (Theorems 2.2 and 2.3) at our disposal, a "from little to big by rescaling" argument deduces the following Montel-type and big Picard-type theorems; see [15] and [17, Theorem 2].

Theorem 2.5. *Let M be an oriented Riemannian n-manifold and N a closed and oriented Riemannian n-manifold, $n \geq 2$. Then a non-normality point $x' \in M$ of a family $\mathcal{F}$ of K-quasiregular mappings, $K \geq 1$, from M to N is contained in $\overline{\bigcup_{f \in \mathcal{F}} f^{-1}(y)}$ for every $y \in N$ except for at most $q'(n,K)$ points.*

Theorem 2.6. *Let M be an oriented Riemannian n-manifold and N a closed and oriented Riemannian n-manifold, $n \geq 2$. Then an essential singularity $x' \in M$ of a K-quasiregular mapping $f : M \setminus \{x'\} \to N$, $K \geq 1$, is accumulated by $f^{-1}(y)$ for every $y \in N$ except for at most $q'(n,K)$ points.*

The similarity Theorems 2.5 and 2.6 goes beyond the statements and we prove these results simultaneously. The argument can also be viewed as a prototype of the proofs of Theorems 1 and 2.

Proof of Theorems 2.5 *and* 2.6. Let $x' \in M$ be either a non-normality point in Theorem 2.5 or an isolated essential singularity in Theorem 2.6.

Let X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$ and let $g \colon X \to N$ be the non-constant quasiregular mapping $v \mapsto f_j \circ \phi^{-1}(x_j + \rho_j v)$ as in Lemma 2.2 or in Lemma 2.3, respectively, associated to this x'. Here $f_j \equiv f$ if x' is as in Lemma 2.6.

Then $g(X)$ is an open subset in N, and satisfies $\#(N \setminus g(X)) \leq q'(n,K)$ by Theorem 2.4.

Let $y \in g(X)$. Fix a subdomain U in N containing y for which some component V of $g^{-1}(U)$ is relatively compact in X. Then $g \colon V \to U$ is proper. By the locally uniform convergence and Lemma 2.1, for every $j \in \mathbb{N}$ large enough, there exists $v_j \in V$ such that $\phi^{-1}(x_j + \rho_j v_j) \in f_j^{-1}(y)$. By the uniform convergence, $\lim_{j \to \infty} \phi^{-1}(x_j + \rho_j v) = x'$ uniformly on $v \in \overline{V}$. Thus $\lim_{j \to \infty} \phi^{-1}(x_j + \rho_j v_j) = x'$ and $x' \in \overline{\bigcup_{j \in \mathbb{N}} f_j^{-1}(y)}$.

Moreover, if x' is an essential singularity of f, then $\phi^{-1}(x_j + \rho_j v_j) \neq x'$ for every $j \in \mathbb{N}$. Thus x' is accumulated by $\bigcup_{j \in \mathbb{N}} f_j^{-1}(y) = f^{-1}(y)$. $\square$

The following Nevanlinna's four totally ramified value theorem is specific to the case $n = 2$. Theorem 2.7 reduces to the original case that $X = \mathbb{R}^2$ and $N = \mathbb{S}^2$ by lifting it to the (conformal) universal coverings of X and N, which are isomorphic to $\mathbb{R}^2$ and a subdomain in $\mathbb{S}^2$, respectively.

Theorem 2.7 (cf. [16, p. 279, Theorem]). *Let $g : X \to N$ be a non-constant quasiregular mapping from X to a closed, oriented and connected Riemannian 2-manifold N, where X is either $\mathbb{R}^2$ or $\mathbb{R}^2 \setminus \{0\}$. Then for every $E \subset N$ containing more than 4 points, $E \cap g(X \setminus B_g) \neq \emptyset$.*

Again, having a Hurwitz-type theorem (Lemma 2.1) and rescaling theorems for both a non-normality point of a family of K-quasiregular mappings and an isolated singularity of a quasiregular mapping (Theorems 2.2 and 2.3) at our disposal, a "from little to big by rescaling" argument deduces the following two big versions of Theorem 2.7.

Lemma 2.8. *Let M be an oriented Riemannian 2-manifold and N a closed and oriented Riemannian 2-manifold, $n \geq 2$. Then a non-normality point $x' \in M$ of a family $\mathcal{F}$ of K-quasiregular mappings, $K \geq 1$, from M to N is contained in $\overline{\bigcup_{f \in \mathcal{F}} (f^{-1}(E) \setminus B_f)}$ for every $E \subset N$ containing more than 4 points.*

Lemma 2.9. *Let M be an oriented Riemannian 2-manifold and N a closed and oriented Riemannian 2-manifold, $n \geq 2$. Then an essential singularity $x' \in M$ of a quasiregular mapping $f\colon M \setminus \{x'\} \to N$ is accumulated by $f^{-1}(E) \setminus B_f$ for every $E \subset N$ containing more than 4 points.*

Again, due the similarity of the statements we give a simultaneous proof.

Proof of Lemmas 2.8 *and* 2.9. Let $x' \in M$ be as in either Lemma 2.8 or Lemma 2.9, and let $g(v) = f_j \circ \phi^{-1}(x_j + \rho_j v)$ be a non-constant quasiregular mapping from X to N as in Lemmas 2.2 and 2.3, respectively, associated to this x', where X is either $\mathbb{R}^2$ or $\mathbb{R}^2 \setminus \{0\}$, and $f_j \equiv f$ in the case that x' is as in Lemma 2.9.

Let E be a subset in N containing more than 4 points. Then by Nevanlinna's four totally ramified values theorem (Theorem 2.7), $g^{-1}(E) \setminus B_g \neq \emptyset$. Fix subdomains U in N intersecting E small enough that some component V of $g^{-1}(U)$ is relatively compact in $X \setminus B_g$. Then $g : V \to U$ is univalent, and by the locally uniform convergence (2.2) or (2.3) on X and the Hurwitz-type theorem (Lemma 2.1), for every $j \in \mathbb{N}$ large enough, there exists $v_j \in V$ such that $\phi^{-1}(x_j + \rho_j v_j) \in f_j^{-1}(E) \setminus B_{f_j}$. Furthermore, $\lim_{j \to \infty} \phi^{-1}(x_j + \rho_j v) = x'$ uniformly on $v \in \overline{V}$. Thus $\lim_{j \to \infty} \phi^{-1}(x_j + \rho_j v_j) = x'$ and $x' \in \overline{\bigcup_{j \in \mathbb{N}} f_j^{-1}(E) \setminus B_{f_j}}$.

Moreover, in the case that x' is as in Lemma 2.9, then $\phi^{-1}(x_j + \rho_j v_j) \neq x'$ for every $j \in \mathbb{N}$, so x' is accumulated by $\bigcup_{j \in \mathbb{N}} f_j^{-1}(E) \setminus B_{f_j} = f^{-1}(E) \setminus B_f$. □

Let $f\colon \Omega \to M$ be a non-constant local uniformly K-quasiregular mapping from an open subset Ω in a closed and oriented Riemannian n-manifold M, $n \geq 2$, to M. The following lemmas are elementary.

Lemma 2.10. *$f^{-1}(\mathcal{E}(f)) \subset \mathcal{E}(f), f^{-1}(D_f) \subset D_f, f(D_f) \subset D_f, f^{-1}(F(f)) \subset F(f), f(F(f)) \subset F(f), f^{-1}(J(f)) \subset J(f)$, and $f(J(f) \cap D_f) \subset J(f)$.*

Proof. The first inclusion $f^{-1}(\mathcal{E}(f)) \subset \mathcal{E}(f)$ is obvious. The inclusion $f^{-1}(D_f) \subset D_f$ immediately follows by the continuity and openness of f. The inclusion $f(D_f) \subset D_f$ also follows by the continuity and openness of f.

The inclusion $f^{-1}(F(f)) \subset F(f)$ follows by the continuity and openness of f and the Arzelà-Ascoli theorem. Indeed, let $x \in f^{-1}(F(f))$. Then $\{f^k; k \in \mathbb{N}\}$ is equicontinuous at $f(x)$, so $\{f^k \circ f; k \in \mathbb{N}\}$ is equicontinuous at x. Hence $x \in F(f)$.

Similarly, the inclusion $f(F(f)) \subset F(f)$ also follows by the continuity and openness of f and the Arzelà-Ascoli theorem. Indeed, let $x \in f(F(f))$, i.e., $x = f(y)$ for some $y \in F(f)$. Then $\{f^k \circ f; k \in \mathbb{N}\}$ is equicontinuous at y, so $\{f^k; k \in \mathbb{N}\}$ is equicontinuous at $x = f(y)$. Hence $x \in F(f)$.

Let us show $f^{-1}(J(f)) \subset J(f)$. The inclusion $f^{-1}(J(f) \setminus D_f) \subset J(f)$ follows from $f(D_f) \subset D_f$, which is equivalent to $f^{-1}(M \setminus D_f) \subset M \setminus D_f$, and $M \setminus D_f \subset J(f)$. The inclusion $f^{-1}(J(f) \cap D_f) \subset J(f)$ follows from $J(f) \cap D_f = D_f \setminus F(f)$ and $f(F(f)) \subset F(f)$.

The final $f(J(f) \cap D_f) \subset J(f)$ follows from $f^{-1}(F(f)) \subset F(f)$, which implies $f(D_f \setminus F(f)) \subset D_f \setminus F(f)$, and $J(f) \cap D_f = D_f \setminus F(f)$. □

Lemma 2.11. *The interior of $J(f) \cap D_f$ is empty unless $J(f) = M$.*

Proof. Let $x \in J(f)$ be an interior point of $J(f)$, and fix an open neighborhood U of x in M contained in $J(f)$. Then by the Montel-type theorem (Theorem 2.5), we have $\#(M \setminus \bigcup_{k\in\mathbb{N}} f^k(U)) < \infty$, so $M = \overline{\bigcup_{k\in\mathbb{N}} f^k(U)}$, which is in $J(f)$ by Lemma 2.10 and the closedness of $J(f)$. □

A cyclic Fatou component of f is a component U of $F(f)$ such that $f^p(U) \subset U$ for some $p \in \mathbb{N}$, which is called a period of U (under f). The proof of the following is almost verbatim to the Euclidean case and we refer to Hinkkanen–Martin–Mayer [9, Proposition 4.9] for the details.

Theorem 2.12. *Let Ω be an open subset in a closed and oriented Riemannian n-manifold M, $n \geq 2$, and $f\colon \Omega \to M$ be a non-elementary local uniformly quasiregular mapping. Then a cyclic Fatou component U of f having a period $p \in \mathbb{N}$ is one of the following:*

(i) a singular (or rotation) domain of f, that is, $f^p\colon U \to f^p(U)$ is univalent and the limit of any locally uniformly convergent sequence $(f^{pk_i})_i$ on U, where $\lim_{i\to\infty} k_i = \infty$, is non-constant,

(ii) an immediate attractive basin of f, that is, the sequence $(f^{pk})_k$ converges locally uniformly on U, the limit is constant, and its value is in U, or

(iii) an immediate parabolic basin of f, that is, the limit of any locally uniformly convergent sequence $(f^{pk_i})_i$ on U, where $\lim_{i\to\infty} k_i = \infty$, is constant and its value is in ∂U.

In the following sections, given a subset S in $\mathbb{R}^n$ and $a, b \in \mathbb{R}$, we denote by $aS + b$ the set $\{av + b \in \mathbb{R}^n; v \in S\}$.

§ 3. Proof of Theorem 1

Let $\mathbb{M}$ be a closed, oriented, and connected Riemannian n-manifold, $n \geq 2$, and $f\colon \mathbb{M} \setminus S_f \to \mathbb{M}$ be a non-constant local uniformly K-quasiregular mapping, $K \geq 1$, where S_f is a countable and closed subset in $\mathbb{M}$ and consists of isolated essential singularities of f and their accumulation points in $\mathbb{M}$.

Lemma 3.1. *The interior of $J(f)$ is empty unless $J(f) = \mathbb{M}$.*

Proof. By Lemma 2.11, the interior of $J(f) \cap D_f$ is empty unless $J(f) = \mathbb{M}$. On the other hand, $J(f) \setminus D_f = \overline{\bigcup_{k\geq 0} f^{-k}(S_f)}$, which is the closure of a countable subset in $\mathbb{M}$, has no interior by the Baire category theorem. □

Set

$$J_1(f) := J(f) \setminus \overline{\bigcup_{k\geq 0} f^{-k}(S_f)} = J(f) \cap D_f \quad \text{and}$$

$$J_2(f) := \bigcup_{k\geq 0} f^{-k}(\{x \in S_f\colon x \text{ is isolated in } S_f\}).$$

The forthcoming arguments in this and the next sections rest on the following observation on the density of $J_1(f) \cup J_2(f)$ in $J(f)$.

Lemma 3.2. *The set $J_1(f) \cup J_2(f)$ is dense in $J(f)$. Furthermore,*

(i) if $\#\bigcup_{k\geq 0} f^{-k}(S_f) < \infty$, then $J_1(f) \cup J_2(f) = J(f)$ and $\#J_2(f) < \infty$;

(ii) if $\#\bigcup_{k\geq 0} f^{-k}(S_f) = \infty$, then $J_1(f) = \emptyset$ and $J(f) = \overline{J_2(f)}$.

Proof. The density in S_f of isolated points of S_f implies $\overline{\bigcup_{k\geq 0} f^{-k}(S_f)} = \overline{J_2(f)}$, so $J_1(f) \cup \overline{J_2(f)} = J(f)$. If $\#\bigcup_{k\geq 0} f^{-k}(S_f) < \infty$, then $J_2(f) = \bigcup_{k\geq 0} f^{-k}(S_f) = \overline{J_2(f)}$, so $J(f) = J_1(f) \cup J_2(f)$ and $\#J_2(f) < \infty$. If $\#\bigcup_{k\geq 0} f^{-k}(S_f) = \infty$, then by the Montel-type theorem (Theorem 2.5), we have $J_1(f) = \emptyset$, so $J(f) = J_1(f) \cup \overline{J_2(f)} = \overline{J_2(f)}$. □

The following is a simple application of the rescaling theorems (Theorems 2.2 and 2.3) to points in the dense subset $J_1(f) \cup J_2(f)$ in $J(f)$. We leave the details to the interested reader.

Lemma 3.3. *Let $a \in J_1(f) \cup J_2(f)$ and let $\phi : D \to \mathbb{R}^n$ be a coordinate chart of $\mathbb{M}$ at a. Then there exist*

(i) sequences (x_m) in $\mathbb{R}^n$ and (ρ_m) in $(0,\infty)$, which respectively tend to $\phi(a)$ and 0 as $m \to \infty$,

(ii) a sequence (k_m) in $\mathbb{N}$, which is constant when $a \in J_2(f)$, and

(iii) a non-constant K-quasiregular mapping $g\colon X \to \mathbb{M}$, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, and $X = \mathbb{R}^n$ when $a \in J_1(f)$,

such that

$$\lim_{m\to\infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v) = g(v) \tag{3.1}$$

locally uniformly on X.

We show the remaining assertions in Theorem 1 in separate lemmas. We continue to use the notation $q'(n,K)$ introduced in Section 2.

We first show both the non-triviality of the Julia set $J(f)$ and the finiteness of the exceptional set $\mathcal{E}(f)$ for non-injective f.

Lemma 3.4. *If $S_f \neq \emptyset$, then f is non-injective, $J(f) \neq \emptyset$, and $\#\mathcal{E}(f) \leq q'(n,K)$. If $S_f = \emptyset$ and f is not injective, then $J(f) \neq \emptyset$, $\mathcal{E}(f) \subset F(f)$, and $\#\mathcal{E}(f) \leq q'(n,K)$.*

Proof. If $S_f \neq \emptyset$, then by the big Picard-type theorem (Theorem 2.6), f is not injective and $\#\mathcal{E}(f) \leq q'(n,K)$, and by the definition of $J(f)$, we have $\emptyset \neq S_f \subset \bigcup_{k\geq 0} f^{-k}(S_f) \subset J(f)$.

From now on, suppose that $S_f = \emptyset$ and $f : \mathbb{M} \setminus S_f \to \mathbb{M}$ is non-injective. Then $\deg f \geq 2$. We show first that $J(f) \neq \emptyset$. Indeed, suppose $J(f) = \emptyset$. Then, by compactness of $\mathbb{M}$, there exists a sequence (k_m) in $\mathbb{N}$ tending to ∞ such that (f^{k_m}) tends to a K-quasiregular endomorphism $h\colon \mathbb{M} \to \mathbb{M}$ uniformly on $\mathbb{M}$. Then for every $m \in \mathbb{N}$ large enough, f^{k_m} is homotopic to h and $\deg h = \deg(f^{k_m}) = (\deg f)^{k_m} \to \infty$ as $m \to \infty$ by the homotopy invariance of the degree. This is a contradiction and $J(f) \neq \emptyset$.

We show now that $\mathcal{E}(f) \subset F(f)$. Let $a \in \mathcal{E}(f)$. Since $\#\bigcup_{k\geq 0} f^{-k}(a) < \infty$, f restricts to a permutation of $\bigcup_{k\geq 0} f^{-k}(a)$. Thus there exists $p \in \mathbb{N}$ for which $f^p(a) = a$ and $i(a, f^p) = \deg(f^p) \geq 2$. Fix a local chart $\phi : D \to \mathbb{R}^n$ at a and identify f^p with

$\phi \circ f^p \circ \phi^{-1}$ in a neighborhood of $a' := \phi(a)$ where the composition is defined. Then there exist a neighborhood U of a' and $C > 0$ such that for every $k \in \mathbb{N}$, f^{pk} is a K-quasiregular mapping from U onto its image, and that for every $k \in \mathbb{N}$ and every $x \in U$,

$$|f^{pk}(x) - f^{pk}(a')| \le C|x - a'|^{(i(a', f^p)^k/K)^{1/(n-1)}}$$

by [18, Theorem III.4.7] (see also [9, Lemma 4.1]). Then $\lim_{k\to\infty} f^{pk} = a'$ locally uniformly on U. Hence $a \in F(f)$.

Finally, we show $\#\mathcal{E}(f) \le q'(n,K)$. If $\#\mathcal{E}(f) > q'(n,K)$, we may fix $A \subset \mathcal{E}(f)$ such that $q'(n,K) < \#A < \infty$ and $A' := \bigcup_{k\ge 0} f^{-k}(A) \subset \mathcal{E}(f)$. Then $q'(n,K) < \#A' < \infty$, and by the above description of each point in $\mathcal{E}(f)$, $f^{-1}(A') = A'$. By $\#A' > q'(n,K)$ and Theorem 2.5, $J(f) \subset \overline{\bigcup_{k\in\mathbb{N}} f^{-k}(A')}$, which contradicts that $\overline{\bigcup_{k\in\mathbb{N}} f^{-k}(A')} = \overline{A'} = A' \subset \mathcal{E}(f) \subset F(f)$. □

We snow next the accumulation of the backward orbits under f of non-exceptional points to $J(f)$ for non-injective f, which implies the perfectness of $J(f)$ for non-elementary f.

Lemma 3.5. *Suppose f is not injective. Then, for every $z \in \mathbb{M} \setminus \mathcal{E}(f)$, each point in $J(f)$ is accumulated by $\bigcup_{k\ge 0} f^{-k}(z)$. Moreover, if f is non-elementary, then $J(f)$ is perfect.*

Proof. Fix $a \in J_1(f) \cup J_2(f)$. Let $g(v) = \lim_{m\to\infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a. Then $\#(\mathbb{M} \setminus g(X)) < \infty$ by Theorem 2.4.

Fix $z \in \mathbb{M} \setminus \mathcal{E}(f)$. Then we can choose subdomains U_1 and U_2 in $g(X)$ intersecting $\bigcup_{k\in\mathbb{N}} f^{-k}(z)$ and having pair-wise disjoint closures so that, for each $i \in \{1,2\}$, some component V_i of $g^{-1}(U_i)$ is relatively compact in X.

For each $i \in \{1,2\}$, $g\colon V_i \to U_i$ is proper. By the locally uniform convergence (3.1) on X and Lemma 2.1, $f^{k_m}(\phi^{-1}(x_m + \rho_m V_i))$ intersects $\bigcup_{k\ge 0} f^{-k}(z)$ for every $m \in \mathbb{N}$ large enough. Thus, for m large enough, we may fix $v_m^{(i)} \in V_i$ satisfying $y_m^{(i)} := \phi^{-1}(x_m + \rho_m v_m^{(i)}) \in \bigcup_{k\ge 0} f^{-k}(z)$.

Let $i \in \{1,2\}$. By the uniform convergence $\lim_{m\to\infty} \phi^{-1}(x_m + \rho_m v) = a$ on $v \in \overline{V_i}$, we have $\lim_{m\to\infty} y_m^{(i)} = a$, and, by the uniform convergence (3.1) on $\overline{V_i}$, we have $\bigcap_{N\in\mathbb{N}} \overline{\{f^{k_m}(y_m^{(i)}); k \ge N\}} \subset g(\overline{V_i}) = \overline{U_i}$. Since $\overline{U_1} \cap \overline{U_2} = \emptyset$, $\{y_m^{(1)}, y_m^{(2)}\} \ne \{a\}$ for $m \in \mathbb{N}$ large enough.

Hence any point $a \in J_1(f) \cup J_2(f)$ is accumulated by $\bigcup_{k\in\mathbb{N}} f^{-k}(z)$, and so is any point in $J(f)$ by Lemma 3.2.

If f is non-elementary, then choosing $z \in J(f) \setminus \mathcal{E}(f)$, we obtain the perfectness of $J(f)$ by the former assertion and $f^{-1}(J(f)) \subset J(f)$. □

We record the following consequence of Lemmas 3.2, 3.4, and 3.5 as a lemma.

Lemma 3.6. *For non-elementary f, $J(f)$ is perfect, $\mathcal{E}(f)$ is finite, and any point in $J(f)$ is accumulated by $(J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$.*

Finally, the following lemma completes the proof of Theorem 1.

Lemma 3.7. *If f is non-elementary, then any point in $J(f)$ is accumulated by the set of all periodic points of f.*

Proof. Fix an open subset U in $\mathbb{M}$ intersecting $J(f)$. Let $a \in (J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$, and let $g(v) = \lim_{m\to\infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a, where X is either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$ and $\phi\colon D \to \mathbb{R}^n$ is a coordinate chart of $\mathbb{M}$ at a. By Lemma 3.5 and Theorem 2.4,

$$(U \cap \bigcup_{k\geq 0} f^{-k}(a)) \cap g(X) \neq \emptyset.$$

Hence we can choose $j_1 \in \mathbb{N} \cup \{0\}$ and a subdomain $D_1 \Subset D$ containing a such that some component U_1 of $f^{-j_1}(D_1)$ is relatively compact in U and that some component V_1 of $g^{-1}(U_1)$ is relatively compact in X. Then $f^{j_1} \circ g : V_1 \to D_1$ is proper.

Choose an open neighborhood $W \Subset X$ of $\overline{V_1}$ small enough that $f^{j_1} \circ g(W) \Subset D$. By the uniform convergence $\lim_{m\to\infty} \phi^{-1}(x_m + \rho_m v) = a \in D_1$ on $v \in \overline{W}$ and the uniform convergence (3.1) on $\overline{W}$, we can define a mapping $\psi : \overline{W} \to \mathbb{R}^n$ and mappings $\psi_m : \overline{W} \to \mathbb{R}^n$ for every $m \in \mathbb{N}$ large enough by

$$\begin{cases} \psi(v) := \phi \circ f^{j_1} \circ g(v) - \phi(a) \quad \text{and} \\ \psi_m(v) := \phi \circ f^{j_1} \circ f^{k_m} \circ \phi^{-1}(x_m + \rho_m v) - (x_m + \rho_m v), \end{cases}$$

so that $\lim_{m\to\infty} \psi_m = \psi$ uniformly on $\overline{W}$.

The limit $\psi\colon V_1 \to \psi(V_1)$ is non-constant, quasiregular, and proper, and satisfies $0 \in \psi(V_1)$ by $a \in D_1 = f^{j_1}(g(V_1))$. Although for each $m \in \mathbb{N}$ large enough, $\psi_m\colon V_1 \to \mathbb{R}^n$ is not necessarily quasiregular, we have $\lim_{m\to\infty} \mu(0, \psi_m, V_1) = \mu(0, \psi, V_1) > 0$ after applying Lemma 2.1 to (ψ_m) and ψ on $\overline{V_1}$. Thus $0 \in \psi_m(V_1)$.

Hence for every $m \in \mathbb{N}$ large enough, there exists $v_m \in V_1$ such that $y_m := \phi^{-1}(x_m + \rho_m v_m)$ is a fixed point of $f^{j_1} \circ f^{k_m}$. Hence also $f^{k_m}(y_m)$ is a fixed point of $f^{j_1} \circ f^{k_m}$. By the uniform convergence (3.1) on $\overline{V_1}$, we have $\bigcap_{N\in\mathbb{N}} \overline{\{f^{k_m}(y_m); k \geq N\}} \subset g(\overline{V_1}) = \overline{U_1} \subset U$, so $f^{k_m}(y_m) \in U$ for every $m \in \mathbb{N}$ large enough.

We conclude that $J(f)$ is in the closure of the set of all periodic points of f, so the perfectness of $J(f)$ completes the proof. $\square$

§ 4. Proof of Theorem 2

Let $\mathbb{M}$ be a closed, oriented, and connected Riemannian n-manifold, $n \geq 2$. Suppose $f\colon \mathbb{M} \setminus S_f \to \mathbb{M}$ is a non-elementary local uniformly K-quasiregular mapping, $K \geq 1$, where S_f is a countable and closed subset in $\mathbb{M}$ and consists of isolated essential singularities of f and their accumulation points in $\mathbb{M}$. We continue to use the notations $J_1(f)$ and $J_2(f)$ introduced in Section 3.

We first show the first assertion of Theorem 2.

Lemma 4.1. *If $F(f)$ is non-empty and connected, then every point in $J(f)$ is accumulated by the set of periodic points of f contained in $J(f)$.*

Proof. By the assumption, $F(f)$ is a fixed cyclic Fatou component of f. We show first that f is not univalent on $F(f)$.

We consider three cases separately. In the case $S_f \neq \emptyset$, by the big Picard-type theorem (Theorem 2.6), for every $y \in F(f)$ except for at most finitely many points, we have $\#f^{-1}(y) = \infty$. In the case that $S_f = \emptyset$ and $B_f \cap F(f) = \emptyset$, we have $\deg f \geq 2$, and also $f(B_f) \cap F(f) = \emptyset$ by $f^{-1}(F(f)) \subset F(f)$. Thus $\#f^{-1}(y) = \deg f \geq 2$ for every $y \in F(f)$. Since $f^{-1}(F(f)) \subset F(f)$, f is not univalent on $F(f)$ in these two cases.

Suppose now that $S_f = \emptyset$ and $B_f \cap F(f) \neq \emptyset$. By the classification of cyclic Fatou components (Theorem 2.12), $F(f)$ is a fixed immediate either attractive or parabolic basin of f. So all the periodic points constructed in Lemma 3.7, but at most one, are in $J(f) = \mathbb{M} \setminus F(f)$. □

Next, we give a useful criterion for the repelling density in $J(f)$.

Lemma 4.2. *Let $a \in (J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$ and suppose that a non-constant quasiregular mapping g in Lemma 3.3 associated to this a satisfies the unramification condition*

$$a \notin \bigcup_{k\in\mathbb{N}} f^k(B_{f^k}) \quad \text{and} \quad J(f) \cap g(X \setminus B_g) \neq \emptyset. \tag{4.1}$$

Then every point in $J(f)$ is accumulated by the set of all repelling periodic points of f.

Proof. Let $a \in (J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$ and let $g(v) = \lim_{m\to\infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a, where $\phi\colon D \to \mathbb{R}^n$ is a coordinate chart of $\mathbb{M}$ at a, and suppose that these a and g satisfy (4.1).

Fix an open subset U in $\mathbb{M}$ intersecting $J(f)$. By Lemma 3.5 and $\#\mathcal{E}(f) < \infty$, there exists $j_1 \in \mathbb{N} \cup \{0\}$ such that $(f^{-j_1}(a) \cap U) \setminus \mathcal{E}(f) \neq \emptyset$. By the latter condition in (4.1), $g(X \setminus B_g)$ is an open subset in $\mathbb{M}$ intersecting $J(f)$. Thus, by Lemma 3.5, there exists $j_2 \in \mathbb{N} \cup \{0\}$ such that $f^{-j_2}((f^{-j_1}(a) \cap U) \setminus \mathcal{E}(f)) \cap g(X \setminus B_g) \neq \emptyset$. Hence by the first condition in (4.1), we can choose a subdomain $D_1 \Subset D \setminus f^{j_1+j_2}(B_{f^{j_1+j_2}})$ containing a such that some component U_1 of $f^{-j_1}(D_1)$ is relatively compact in U and that some component V_1 of $g^{-1}(f^{-j_2}(U_1))$ is relatively compact in $X \setminus B_g$. Then $f^{j_1+j_2} \circ g : V_1 \to D_1$ is univalent.

By the same argument as in the proof of Lemma 3.7, we may choose, for every $m \in \mathbb{N}$ large enough, a point $v_m \in V_1$ such that $y_m := \phi^{-1}(x_m + \rho_m v_m)$ is a fixpoint of $f^{j_1+j_2} \circ f^{k_m}$. By the uniform convergence (3.1) on $\overline{V_1}$, we have $\bigcap_{N\in\mathbb{N}} \overline{\{f^{j_2} \circ f^{k_m}(y_m); k \geq N\}} \subset f^{j_2}(g(\overline{V_1})) = \overline{U_1} \subset U$. Thus $f^{j_2} \circ f^{k_m}(y_m) \in U$ for every $m \in \mathbb{N}$ large enough.

Moreover, by the locally uniform convergence (3.1) on X and Lemma 2.1, the mapping $v \mapsto f^{j_1+j_2} \circ f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ is a univalent mapping from V_1 onto its image for every $m \in \mathbb{N}$ large enough. Hence

$$f^{j_1+j_2} \circ f^{k_m} : \phi^{-1}(x_m + \rho_m V_1) \to f^{j_1+j_2} \circ f^{k_m}(\phi^{-1}(x_m + \rho_m V_1))$$

is univalent for $m \in \mathbb{N}$ large enough. By the uniform convergence

$$\lim_{m\to\infty} \phi^{-1}(x_m + \rho_m v) = a \in D_1 = f^{j_1+j_2} \circ g(V_1)$$

on $v \in \overline{V_1}$ and the uniform convergence (3.1) on $\overline{V_1}$,

$$\phi^{-1}(x_m + \rho_m V_1) \Subset f^{j_1+j_2} \circ f^{k_m}(\phi^{-1}(x_m + \rho_m V_1))$$

for every $m \in \mathbb{N}$ large enough. Hence for every $m \in \mathbb{N}$ large enough, y_m is a repelling fixed point of $f^{j_1+j_2} \circ f^{k_m}$.

We conclude that $J(f)$ is in the closure of the set of all repelling periodic points of f, so the perfectness of $J(f)$ completes the proof. $\square$

We show the latter assertion of Theorem 2 under the conditions given there, separately.

Condition (i). Suppose $\#\bigcup_{k\geq 0} f^{-k}(S_f) < \infty$. Then by Lemmas 3.2 and 3.4, we have $\#(J_2(f) \cup \mathcal{E}(f)) < \infty$ and $J_1(f) = J(f) \setminus J_2(f)$. Suppose also that $\dim J(f) \geq n-1$. For every $k \in \mathbb{N}$, $\dim f^k(B_{f^k}) \leq n-2$, and then $\dim(\bigcup_{k\in\mathbb{N}} f^k(B_{f^k})) \leq n-2$ ([11, §2.2, Theorem III]). Hence we can fix $a \in J(f) \setminus (J_2(f) \cup \mathcal{E}(f) \cup \bigcup_{k\in\mathbb{N}} f^k(B_{f^k})) = J_1(f) \setminus (\mathcal{E}(f) \cup \bigcup_{k\in\mathbb{N}} f^k(B_{f^k}))$, and let $g\colon \mathbb{R}^n \to \mathbb{M}$ be a non-constant quasiregular mapping as in Lemma 3.3 associated to this a. Then $\dim g(B_g) \leq n-2$, so $J(f) \cap g(\mathbb{R}^n \setminus B_g) \neq \emptyset$.

The unramification condition (4.1) is satisfied by these a and g, and Lemma 4.2 completes the proof in this case.

Condition (ii). Let a be a repelling periodic point of f having a period $p \in \mathbb{N}$ in $D_f \setminus (\mathcal{E}(f) \cup \bigcup_{k\in\mathbb{N}} f^k(B_{f^k}))$. Then $a \in (J(f) \setminus \mathcal{E}(f)) \cap D(f) = J_1(f) \setminus \mathcal{E}(f)$. Let

$g(v) = \lim_{m\to\infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from $\mathbb{R}^n$ to $\mathbb{M}$ as in Lemma 3.3 associated to this a, where $\phi\colon D \to \mathbb{R}^n$ is a coordinate chart of $\mathbb{M}$ at this a. By [9, Theorem 6.3], we may, in fact, assume that $x_m \equiv \phi(a)$ and $p|k_m$ for all $m \in \mathbb{N}$, and g is in this case usually called a *Koenigs mapping of* f^p *at* a. Then $g(0) = a$, and by the proof of [9, Theorem 6.3], we also have $0 \notin B_g$. Hence $a \in J(f) \cap g(\mathbb{R}^n \setminus B_g)$, and (4.1) is satisfied by these a and g. Lemma 4.2 completes the proof in this case.

Condition (iii). Suppose that $J(f) \not\subset \bigcap_{j\in\mathbb{N}} \overline{\bigcup_{k\geq j} f^k(B_{f^k})}$. By the closedness of $\bigcap_{j\in\mathbb{N}} \overline{\bigcup_{k\geq j} f^k(B_{f^k})}$ and Lemma 3.6, we indeed have $J(f) \not\subset (\mathcal{E}(f) \cup \bigcap_{j\in\mathbb{N}} \overline{\bigcup_{k\geq j} f^k(B_{f^k})})$. Hence we can fix $N \in \mathbb{N}$ so large that the open subset $U_N := \mathbb{M} \setminus (\mathcal{E}(f) \cup \overline{\bigcup_{k\geq N} f^k(B_{f^k})})$ in $\mathbb{M}$ intersects $J(f)$.

Let $a \in (J_1(f) \cup J_2(f)) \cap U_N \subset (J_1(f) \cup J_2(f)) \setminus \mathcal{E}(f)$, and let $g(v) = \lim_{m\to\infty} f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a. Then $\#(\mathbb{M} \setminus g(X)) < \infty$ by Theorem 2.4. We claim that $\# \bigcup_{k\geq N} f^{-k}(a) = \infty$. Indeed, in the case $\# \bigcup_{k=0}^{N-1} f^{-k}(a) < \infty$, this follows by $a \notin \mathcal{E}(f)$. In the case $\# \bigcup_{k=0}^{N-1} f^{-k}(a) = \infty$, we have $S_f \neq \emptyset$. By applying the big Picard-type theorem (Theorem 2.6) in at most N times, we obtain $\# f^{-N}(a) = \infty$. Hence we can fix $j_1 \geq N$ such that $f^{-j_1}(a) \cap g(X) \neq \emptyset$, and a subdomain $U \Subset U_N$ containing a so small that some component V of $(f^{j_1} \circ g)^{-1}(U)$ is relatively compact in X. Then $g\colon V \to g(V)$ is proper.

By the uniform convergence (3.1) on $\overline{V}$, for every $m \in \mathbb{N}$ large enough, $f^{j_1} \circ f^{k_m} \circ \phi^{-1}(x_m + \rho_m V) \Subset U_N$. Then by $j_1 \geq N$ and the definition of U_N, $f^{k_m} : \phi^{-1}(x_m + \rho_m V) \to f^{k_m}(\phi^{-1}(x_m + \rho_m V))$ is univalent, so the mapping $v \mapsto f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ from V onto its image is univalent. Hence by the locally uniform convergence (3.1) on X and the Hurwitz-type theorem (Lemma 2.1), $V \cap B_g = \emptyset$. Then $\emptyset \neq f^{-j_1}(a) \cap g(V) \subset J(f) \cap g(X \setminus B_g)$, and (4.1) is satisfied by these a and g. Lemma 4.2 completes the proof in this case.

Condition (iv). Suppose that $n = 2$. If $\# \bigcup_{k\geq 0} f^{-k}(S_f) < \infty$, then by Lemmas 3.2 and 3.6, $J_1(f) = J(f) \setminus J_2(f)$ is uncountable. Since $\#\mathcal{E}(f) < \infty$ (in Lemma 3.6) and $\bigcup_{k\geq 0} B_{f^k}$ is countable (when $n = 2$), we may fix $a \in J_1(f) \setminus (J_2(f) \cup \mathcal{E}(f) \cup \bigcup_{k\in\mathbb{N}} f^k(B_{f^k})) \subset J_1(f) \setminus \mathcal{E}(f)$. Let $g\colon \mathbb{R}^n \to \mathbb{M}$ be a non-constant quasiregular mapping as in Lemma 3.3 associated to this a. By the countability of B_g (when $n = 2$) and the uncountability of $g^{-1}(J(f))$, we also have $g^{-1}(J(f)) \not\subset B_g$. The unramification condition (4.1) is satisfied by these a and g, and Lemma 4.2 completes the proof in this case.

In the remaining case $\# \bigcup_{k\geq 0} f^{-k}(S_f) = \infty$, the argument similar to the above does not work. For $n = 2$, instead of Lemma 4.2, we rely on the big versions (Lemmas 2.8 and 2.9) of the Nevanlinna four totally ramified value theorem (Theorem 2.7) to show Theorem 2 under $n = 2$, which is independent of the above proof specific to the case $\#\bigcup_{k\geq 0} f^{-k}(S_f) < \infty$.

Proof of Theorem 2 *under* $n = 2$. Set

$$J'(f) := \begin{cases} J_1(f) \setminus \{\text{all periodic points of } f\} & \text{if } \#\bigcup_{k\geq 0} f^{-k}(S_f) < \infty, \\ J_2(f) & \text{if } \#\bigcup_{k\geq 0} f^{-k}(S_f) = \infty. \end{cases}$$

We claim that $J'(f)$ is dense in $J(f)$. If $\#\bigcup_{k\geq 0} f^{-k}(S_f) = \infty$, we have $J(f) = \overline{J_2(f)} = \overline{J'(f)}$ by Lemma 3.2. Thus we may assume that $\#\bigcup_{k\geq 0} f^{-k}(S_f) < \infty$ and it suffices to show that $J(f) = \overline{J'(f)}$.

By Lemmas 3.2 and 3.6, the set $J_1(f)$ is uncountable. Since f has at most countably many periodic points, $J'(f)$ is non-empty. Let $y \in J'(f)$. If $J(f) \not\subset \overline{J'(f)}$, then every point in $J(f) \setminus \overline{J'(f)}$ is accumulated by $\bigcup_{k\geq 0} f^{-k}(y)$ by Lemma 3.5. On the other hand, by Lemma 3.2, $\#J_2(f) < \infty$. Since $J_1(f) = J(f) \setminus J_2(f)$, there exists $x \in \bigcup_{k\geq 0} f^{-k}(y) \cap (J_1(f) \setminus \overline{J'(f)})$. Thus x is a periodic point of f, and so is y, which is a contradiction. Hence $J(f) = \overline{J'(f)}$ in the case $\#\bigcup_{k\geq 0} f^{-k}(S_f) < \infty$.

Since $J(f)$ is perfect, $\#J'(f) = \infty$. Fix an open subset U in $\mathbb{M}$ intersecting $J(f)$. We claim that there exists $a \in J'(f)$ such that $\#(U \cap \bigcup_{k\geq 0}(f^{-k}(a) \setminus B_{f^k})) = \infty$. Indeed, let $E \subset J'(f)$ such that $4 < \#E < \infty$ and let $b' \in U \cap (J_1(f) \cup J_2(f))$. For $b' \in J_1(f)$, $\{f^k; k \geq N\}$ is not normal at b' for any $N \in \mathbb{N}$. Hence $b' \in \bigcap_{N\in\mathbb{N}} \overline{\bigcup_{k\geq N}(f^{-k}(E) \setminus B_{f^k})}$ by Lemma 2.8. Moreover, if $b' \in f^{-k}(E)$ for infinitely many $k \in \mathbb{N}$, then, by $\#E < \infty$, $f^{k_1}(b') = f^{k_2}(b') \in E$ for some $k_1 < k_2$. Thus $f^{k_1}(b') \in E$ is a periodic point of f, which contradicts $E \subset J'(f)$. Hence b' is accumulated by $\bigcup_{k\geq 0}(f^{-k}(E) \setminus B_{f^k})$. In the case $b' \in J_2(f)$, b' is an isolated essential singularity of f^{j_1} for some $j_1 \in \mathbb{N}$, so by Lemma 2.9, b' is accumulated by $f^{-j_1}(E) \setminus B_{f^{j_1}}$. In both cases, by $\#E < \infty$, we can choose $a \in E$ such that $\#(U \cap \bigcup_{k\geq 0}(f^{-k}(a) \setminus B_{f^k})) = \infty$.

Let $g(v) = f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ be a non-constant quasiregular mapping from X to $\mathbb{M}$ as in Lemma 3.3 associated to this a, where X is either $\mathbb{R}^2$ or $\mathbb{R}^2 \setminus \{0\}$ and $\phi : D \to \mathbb{R}^2$ is a coordinate chart of $\mathbb{M}$ at a. Then by the Nevanlinna four totally ramified value theorem (Theorem 2.7),

$$\left(U \cap \bigcup_{k\geq 0} (f^{-k}(a) \setminus B_{f^k}) \right) \cap g(X \setminus B_g) \neq \emptyset.$$

Hence we can choose $j_1 \in \mathbb{N} \cup \{0\}$ and a subdomain $D_1 \Subset D$ containing a such that some component U_1 of $f^{-j_1}(D_1)$ is relatively compact in $U \setminus B_{f^{j_1}}$ and that some component V_1 of $g^{-1}(U_1)$ is relatively compact in $X \setminus B_g$. Then $f^{j_1} \circ g : V_1 \to D_1$ is univalent.

By the same argument in the proof of Lemma 3.7, for every $m \in \mathbb{N}$ large enough, we can choose $v_m \in V_1$ such that $y_m := \phi^{-1}(x_m + \rho_m v_m)$ is a fixed point of $f^{j_1} \circ f^{k_m}$, and so is $f^{k_m}(y_m)$, and we also have $f^{k_m}(y_m) \in U$ for every $m \in \mathbb{N}$ large enough.

Moreover, by the locally uniform convergence (3.1) on X and Lemma 2.1, the mapping $v \mapsto f^{j_1} \circ f^{k_m} \circ \phi^{-1}(x_m + \rho_m v)$ is also a univalent mapping from V_1 onto its image for every $m \in \mathbb{N}$ large enough. Hence

$$f^{j_1} \circ f^{k_m} : \phi^{-1}(x_m + \rho_m V_1) \to f^{j_1} \circ f^{k_m}(\phi^{-1}(x_m + \rho_m V_1))$$

is univalent for $m \in \mathbb{N}$ large enough. By the uniform convergence $\lim_{m\to\infty} \phi^{-1}(x_m + \rho_m v) = a \in D_1 = f^{j_1} \circ g(V_1)$ on $v \in \overline{V_1}$ and the uniform convergence (3.1) on $\overline{V_1}$,

$$\phi^{-1}(x_m + \rho_m V_1) \Subset f^{j_1} \circ f^{k_m}(\phi^{-1}(x_m + \rho_m V_1)).$$

for every $m \in \mathbb{N}$ large enough. Hence y_m is a repelling fixed point of $f^{j_1} \circ f^{k_m}$ for every $m \in \mathbb{N}$ large enough.

We conclude that $J(f)$ is in the closure of the set of all repelling periodic points of f, so the perfectness of $J(f)$ completes the proof. □

§ 5. On the non-injectivity and non-elementarity of f

In the setting of Theorem 1, we have the following result on the non-elementarity of non-injective UQR-mappings.

Lemma 5.1. *Let $\mathbb{M}$ and $f\colon \mathbb{M} \setminus S_f \to \mathbb{M}$ be as in Theorem 1. Suppose in addition that f is non-injective. Then f is non-elemenatary if either $S_f = \emptyset$ or $\#\bigcup_{k\geq 0} f^{-k}(S_f) > q'(n,K)$.*

Proof. For $S_f = \emptyset$ the claim follows from Theorem 1. Suppose $\#\bigcup_{k\geq 0} f^{-k}(S_f) > q'(n,K)$. By the big Picard-type theorem (Theorem 2.6), we have $\#\bigcup_{k\geq 0} f^{-k}(S_f) = \infty$. Thus, by Lemma 3.2, $J(f) = \overline{\bigcup_{k\geq 0} f^{-k}(S_f)}$. Hence $J(f) \not\subset \mathcal{E}(f)$ since $\#\mathcal{E}(f) < \infty$. □

It seems an interesting problem whether a non-injective f is always non-elementary. This is the case in holomorphic dynamics, i.e., the case that $\mathbb{M} = \mathbb{S}^2$ and $K = 1$. Indeed, if $0 < \#\bigcup_{k\geq 0} f^{-k}(S_f) \leq q'(2,1) = 2$, f can be normalized to be either a transcendental entire function on $\mathbb{C}$ or a holomorphic endomorphism of $\mathbb{C} \setminus \{0\}$ having essential singularities at $0, \infty$, both of which are known to be non-elementary.

References

[1] I. Baker, *Repulsive fixpoints of entire functions*, Math. Z. **104** (1968), 252–256.

[2] D. Bargmann, *Simple proofs of some fundamental properties of the Julia set*, Ergodic Theory Dynam. Systems **19** (1999), 553–558.

[3] F. Berteloot and J. Duval, *Une démonstration directe de la densité des cycles répulsifs dans l'ensemble de Julia*, Complex analysis and geometry (Paris, 1997), volume 188 of *Progr. Math.*, pages 221–222. Birkhäuser, Basel, 2000.

[4] P. Bhattacharyya, *Iteration of analytic functions*, PhD thesis, Imperial College London (University of London), 1969.

[5] A. Bolsch, *Repulsive periodic points of meromorphic functions*, Complex Variables Theory Appl. **31** (1996), 75–79.

[6] A. N. Fletcher and D. A. Nicks, *Julia sets of uniformly quasiregular mappings are uniformly perfect*. Math. Proc. Cambridge Philos. Soc. **151** (2011), 541–550.

[7] J. Heinonen and S. Rickman, *Geometric branched covers between generalized manifolds*, Duke Math. J. **113** (2002), 465–529.

[8] M. E. Herring, *An extension of the Julia-Fatou theory of iteration*, Ph.D. thesis, London, 1995.

[9] A. Hinkkanen, G. J. Martin, and V. Mayer, *Local dynamics of uniformly quasiregular mappings*, Math. Scand. **95** (2004), 80–100.

[10] I. Holopainen and S. Rickman, *Ricci curvature, Harnack functions, and Picard type theorems for quasiregular mappings*, Analysis and topology, pages 315–326. World Sci. Publ., River Edge, NJ, 1998.

[11] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Mathematical Series, v. 4, Princeton University Press, Princeton, N. J., 1941.

[12] T. Iwaniec and G. Martin, *Quasiregular semigroups*, Ann. Acad. Sci. Fenn. **21** (1996), 241–254.

[13] ———, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2001.

[14] J. Milnor, *Dynamics in one complex variable*, volume 160 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, third edition, 2006.

[15] R. Miniowitz, *Normal families of quasimeromorphic mappings*, Proc. Amer. Math. Soc. **84** (1982), 35–43.

[16] R. Nevanlinna, *Analytic functions*, Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162. Springer-Verlag, New York, 1970.

[17] Y. Okuyama and P. Pankka, *Rescaling principle for isolated essential singularities of quasiregular mappings*, ArXiv e-prints, Nov. 2012.

[18] S. Rickman, *Quasiregular mappings*, volume 26 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993.

[19] W. Schwick, *Repelling periodic points in the Julia set*, Bull. London Math. Soc. **29** (1997), 314–316.

[20] H. Siebert, *Fixpunkte und normale Familien quasiregulärer Abbildungen*, Dissertation, University of Kiel, available at http://e-diss.uni-kiel.de/diss_1260. 2004.

[21] ———, *Fixed points and normal families of quasiregular mappings*, J. Anal. Math. **98** (2006), 145–168.

[22] J. Väisälä, *Discrete open mappings on manifolds*, Ann. Acad. Sci. Fenn. Ser. A I , No. 392:10, 1966.